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TIME-PERIODIC SOLUTION OF THE SYSTEM OF BOUNDARY LAYER EQUATIONS

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The system of boundary layer equations of unsteady axisymmetric flow of incompressible fluid in the presence of blowing or suction through the boundary surface of a body is considered. Proof is given of the existence and uniqueness of a time-periodic solution of such system in the neighborhood of the critical point (forced oscillations), when the external flow is periodic with respect to time and the functions defining the body shape and the blowing or suction conditions are known. This problem was considered in detail in [1] for specific initial conditions (as a whole dependent on t), where, in particular, data on the stability of such flows are presented.

Let us consider the system of equations

$$u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \quad p_y = 0, \quad (ru)_x + (rv)_y = 0 \tag{1}$$

for the region $D\ \{-\infty < t < +\infty,\ 0 \leqslant x \leqslant X,\ 0 \leqslant y < \infty\}$ with boundary conditions

$$u\,|_{x=0} = 0, \quad u\,|_{y=0} = 0, \quad v\,|_{y=0} = v_0(t, x), \quad u \to U(t, x) \quad \text{for} \quad y \to \infty \tag{2}$$

$$u(t+T,\ x,\ y) = u(t,\ x,\ y)$$

where u and v are velocity components parallel and normal to the wall of the body; $U(t, x)$ is the longitudinal component of the external flow velocity, with $U(t, 0) = 0$, and $U(t, x) > 0$ for $x > 0$; $-p_x = U_t + UU_x$, ν is the viscosity coefficient (for density $\rho \equiv 1$); $r(t, x)$ is the distance of point x on the body surface from the latter axis of symmetry; and $r(t, 0) = 0$ and $r(t, x) > 0$ for $x > 0$. We assume that in the region D, $U_x > 0$, $r_x > 0$ and $p_x / U < 0$. Let U, r, p and v_0 be a specified periodic functions with respect to t of period T.

To analyze the problem (1), (2) we introduce new independent variables

$$\tau = t, \qquad \xi = x, \qquad \eta = \frac{u(t, x, y)}{U(t, x)} \tag{3}$$

We then obtain for function $w = u_y / U$ in region $\Omega\ \{-\infty < \tau < +\infty,\ 0 \leqslant \xi \leqslant X,\ 0 \leqslant \eta < 1\}$ the equation

$$\nu w^2 w_{\eta\eta} - w_\tau - \eta U w_\xi + A w_\eta + B w = 0 \tag{4}$$

with boundary and periodicity conditions

$$w\,|_{\eta=1} = 0, \qquad (\nu w w_\eta - v_0 w + C)\,|_{\eta=0} = 0 \tag{5}$$

$$w(\tau + T, \xi, \eta) = w(\tau, \xi, \eta)$$

where

$$A = (\eta^2 - 1) U_x + (\eta - 1) \frac{U_t}{U}, \quad B = \eta \left(r_x \frac{U}{r} - U_x \right) - \frac{U_t}{U}$$

$$C = - \frac{p_x}{U} = U_x + \frac{U_t}{U}$$

The unknown function $u\ (t,\ x,\ y)$ is defined by the equalities

$$y = \int_0^{\eta} \frac{ds}{w\,(t,\ x,\ s)}, \qquad \eta = \frac{u\ (t,\ x,\ y)}{U\ (t,\ x)}$$

and function $v\ (t,\ x,\ y)$ is determined by the first equation of system (1) [1].

Let us assume that A, B, C and v_0 and their derivatives with respect to t and x are bounded. Using the method of straight lines [1], we shall prove on suitable assumptions the existence and the uniqueness of the solution of problem (4), (5) and obtain, as the corollary, the related theorems on the periodicity with respect to t of solutions of the input problem (1), (2).

Let $f^{m,k}(\eta) \equiv f\ (mh_1,\ kh_2,\ \eta)$ for any function $f\ (\tau,\ \xi,\ \eta)$, h_1 and $h_2 =$ const > 0. We substitute for Eq. (4) with conditions (5) the following system of differential equations:

$$L_{m,k}(w) \equiv \nu\, (w^{m,k})^2 w_{\eta\eta}^{m,k} - \frac{w^{m,k} - w^{m-1,k}}{h_1} -$$

$$- (\eta U^{m,k} + \mu_k h_2^{\gamma}) \frac{w^{m,k} - w^{m,k-1}}{h_2} + A^{m,k} w_{\eta}^{m,k} + B^{m,k} w^{m,k} = 0 \qquad (6)$$

$$m = 1, \ldots, N; \quad N = T / h_1; \quad k = 0, 1, \ldots, l; \quad l = [X / h_2]; \quad 0 \leqslant \eta < 1$$

$$U^{m,0} = 0, \quad \mu_0 = 0, \quad \mu_k = \text{const} > 0, \quad (k \geqslant 1), \quad \gamma = \text{const}, \quad 0 < \gamma < 1$$

(where h_1 is such that $T\ /\ h_1$ is an integer) with boundary conditions

$$w^{m,k}(1) = 0, \quad \lambda_{m,k}(w) \equiv (\nu w^{m,k} w_{\eta}^{m,k} - v_0^{m,k} w^{m,k} + C^{m,k})|_{\eta=0} = 0 \qquad (7)$$

and condition of periodicity

$$w^{0,k}(\eta) = w^{N,k}(\eta) \qquad (8)$$

We denote by M_j, E_j and α positive constants independent of h_1 and h_2.

Lemma 1. The system of differential equations (6) with conditions (7) and (8) has the solution $w^{m,k}(\eta)$ $(0 \leqslant m \leqslant N,\ 0 \leqslant k \leqslant l)$, which is continuous for $0 \leqslant \eta \leqslant 1$ and has all derivatives for $0 \leqslant \eta < 1$. The estimate

$$M_1 (1 - \eta) \leqslant w^{m,k}(\eta) \leqslant M_2 (1 - \eta)\, \sigma \qquad (9)$$

$$\sigma = \sqrt{- \ln \mu (1 - \eta)} \ \text{ for } \ kh_2 \leqslant X,\ h_i \leqslant h_0 = \text{const} > 0, \quad \mu = \text{const}, \quad \mu \in (0, \mu^0)$$

where μ^0 is a certain constant defined by the input data of problem (1), (2), is valid for this solution.

Proof. We derive the solution of system (6) with conditions (7) and (8) as the limit of solution of system

$$L_{m,k}^{\varepsilon}(w) \equiv \varepsilon w_{\eta\eta\eta}^{m,k} + L_{m,k}(w) = 0 \quad \text{for} \quad \varepsilon \to 0 \qquad (10)$$

$$m = 1, 2, \ldots, N; \ k = 0, 1, \ldots, l; \ \varepsilon > 0, \ 0 \leqslant \eta < 1$$

with conditions (7) and (8). The proof of Lemma 1 is to a great extent similar to that of Lemmas 3 and 7 in [1].

Let us examine functions

$$V_1(\xi, \eta) = M_3(1-\eta)\exp(-\alpha\xi)$$

$$V_2(\xi, \eta) = M_4(1-\eta)\sigma \quad (M_3, \alpha, M_4 = \text{const} > 0).$$

As shown in [1], constants M_3, α, M_4 and μ° can be chosen so as to satisfy the following inequalities:

$$L^\varepsilon_{m,k}(V_1) \geqslant 0, \quad \lambda_{m,k}(V_1) > 0, \quad L^\varepsilon_{m,k}(V_2) \leqslant 0, \quad \lambda_{m,k}(V_2) < 0 \tag{11}$$

Note that M_3, M_4, α and μ° are independent of ε, h_1 and h_2.

Proof of the existence of solution of the problem defined by (10), (7) and (8) is based on the Schauder theorem [2]. Let S be a set of bounded vector functions $\theta = (\theta^0, \theta^1 \ldots \theta^l)$ such that

$$V_1(kh_2, \eta) \leqslant \theta^k(\eta) \leqslant V_2(kh_2, \eta), \; k = 0, 1, \ldots, l; \; l = [X/h_2] \tag{12}$$

Let us examine the shift operator R which associates the vector function θ to these vector functions, $w^N = (w^{N,0}, \ldots, w^{N,l})$, where $w^{N,k} = w^{m,k}$ for $m = N = T/h_1$, and $w^{m,k}$ is the solution of system (10) with boundary conditions (7) and initial condition

$$w^{0,k} = \theta^k \qquad (0 \leqslant k \leqslant l) \tag{13}$$

The operator R according to Lemma 7 in [1], is determinate on set S. Functions $w^{N,k}(\eta)$ are continuous for $0 \leqslant \eta \leqslant 1$, have finite derivatives for $0 \leqslant \eta \leqslant 1$, and for certain positive constants E_1 and E_2 independent of ε, h_1 and h_2 the estimate

$$E_1(1-\eta) \leqslant w^{N,k}(\eta) \leqslant E_2(1-\eta)\sigma$$

is valid.

The operator R maps S into itself. This statement is implied by the inequalities (11) and (12), since the estimate

$$V_1^{m,k}(\eta) \leqslant w^{m,k}(\eta) \leqslant V_2^{m,k}(\eta) \tag{14}$$

is valid for the solution of the problem defined by (10), (7) and (13), while $V_i^{N,k} = V_i^{0,k} (i = 1, 2)$. Proof of the estimate (14) is obtained by the principle of maximum (see Lemma 3 in [1]).

The set RS is compact in S, since the first and second derivatives $w^{m,k}(\eta)$ are bounded by a constant which depends on the input data of problem (1), (2), ε, h_1, h_2 and on functions V_1 and V_2. This follows directly from the first-order equations derived from system (10) which are satisfied by $w^{m,k}_\eta$ and from the estimate of $w^{m,k}_\eta$ for $\eta = 0$. The latter follows from the boundary condition (7) (we assume here that ε and h_i are fixed, and the estimate $w^{m,k}_\eta$ depends on ε and h_i); the estimate is uniform with respect to ε for $0 \leqslant \eta \leqslant (1-\delta)$ where $\delta > 0$. The derivative $w^{m,k}_{\eta\eta}$ is defined by (10).

The continuity of operator R is implied by the equations and boundary conditions which are satisfied by various solutions of the problem (10), (7), (13) corresponding to various θ, as well as by the estimates of these solutions and their derivatives.

We thus conclude that the absolutely continuous shift operator R maps the bounded closed convex set S of the space of bounded functions into itself. Hence, by the Schauder theorem [2] there exists a stationary point $\theta_0 = (\theta_0^0, \theta_0^1 \ldots, \theta_0^l)$ which is the image of R, i.e., $R\theta_0 = \theta_0$. This equality implies that $\theta_0 \in RS$, hence $\theta_{0\eta\eta}$ is bounded. The sought periodic solution $w^{m,k}(\eta)$ of problem (10), (7) is derived as the solution of system (10) with boundary conditions (7) and the initial condition $w^{0,k} = \theta_0^k$. Differentiating Eqs. (10) with respect to η, we find that $w^{m,k}_{\eta\eta\eta}$ $(m \geqslant 1)$ and, consequently,

$w^{0,k}_{\eta\eta\eta}$ are bounded. Repeating this process, we come to the conclusion that $w^{m,k}(\eta)$ are infinitely differentiable functions of η, and that the derivatives $\partial_\eta{}^j w^{m,k}$ $(j = 1, 2, \ldots)$ are uniformly bounded with respect to ε along the segment $0 \leqslant \eta \leqslant (1 - \delta)$, where δ is an arbitrary positive number.

Let us now find the solution of problem (6) – (8). By the Arzela theorem it is possible to select from the set $w_\varepsilon^{m,k}$, of solutions of the problem (10), (7), (8) a sequence $w_{\varepsilon_n}^{m,k}$ which is uniformly convergent together with its derivatives along any segment $0 \leqslant \eta \leqslant (1 - \delta)$ for $\varepsilon_n \to 0$. Since M_3, M_4, α and μ° have been assumed independent of ε, h_1 and h_2, the estimates

$$M_3 (1 - \eta) e^{-\alpha k h_2} \leqslant w^{m,k}(\eta) \leqslant M_4 (1 - \eta) \sigma$$

which are uniform with respect to h_1 and h_2, are valid for functions $w^{m,k}$. It follows from these inequalities $w^{m,k}(1) = 0$ and $w^{m,k}(\eta)$ are continuous for $0 \leqslant \eta \leqslant 1$, and for $\eta < 1$; $w^{m,k}(\eta)$ satisfy system (6) and conditions (7) and (8). Lemma 1 is proved.

Additional assumptions with respect to function $U(t, x)$ make it possible to improve the lower accuracy limit of $w^{m,k}(\eta)$.

Lemma 2. Let $(U_t / U)|_{\xi=0} = 0$. Then for $0 \leqslant \eta \leqslant 1$, $0 \leqslant k h_2 \leqslant X$ and $0 < \mu < \mu^\circ$ $(\mu^\circ < 1/\sqrt{e})$ the estimate

$$w^{m,k}(\eta) \geqslant M_5 (1 - \eta) \sigma \tag{15}$$

is valid for the solution of problem (6) – (8).

Proof. Let $V_3^{m,k} = M_6 (1 - \eta) \sigma e^{-\alpha k h_2}$. Then

$$L_{m,k}(V_3) = V_3^{m,k} \Big\{ -\nu M_6^2 e^{-2\alpha k h_2} \left(\frac{1}{2} + \frac{1}{4\sigma^2} \right) + (1 + \eta) U_x^{m,k} \left(1 - \frac{1}{2\sigma^2} \right) - \\ - \frac{1}{2\sigma^2} \left(\frac{U_t}{U} \right)^{m,k} + \eta \left(r_x \frac{U}{r} - U_x \right)^{m,k} + \alpha (\eta U^{m,k} + \mu_k h_2^\gamma) e^{\alpha h_2'} \Big\} > 0$$

for $\eta < 1$ and for reasonably great α and reasonably small M_6 and μ°, since $U_x > 0$, $|(r_x U)/r - U_x| \leqslant \alpha U$ and $\sigma^{-2} \leqslant \sigma^{-2}(0) \to 0$ for $\mu \to 0$; $0 < h_2' < h_2$. Let $\lambda^1_{m,k}(w) \equiv \lambda_{m,k}(w) / w^{m,k}(0)$. We have

$$\lambda^1_{m,k}(V_3) = \left[-\nu M_6 e^{-\alpha k h_2} \sigma \left(1 - \frac{1}{2\sigma^2} \right) - v_0^{m,k} + \frac{C^{m,k}}{M_6 e^{-\alpha k h_2} \sigma} \right] \Bigg|_{\eta=0} > 0$$

provided M_6 is reasonably small, since $C > 0$

Let us consider functions $y^{m,k} = (V_3^{m,k} - w^{m,k}) e^{-\beta k h_2}$. We shall prove that $y^{m,k} \leqslant 0$. For $y^{m,k}$ we obtain the inequalities

$$[L_{m,k}(V_3) - L_{m,k}(w)] e^{-\beta k h_2} = \nu (w^{m,k})^2 y^{m,k}_{\eta\eta} - \frac{y^{m,k} - y^{m-1,k}}{h_1} - \\ - (\eta U^{m,k} + \mu_k h_2^\gamma) \frac{y^{m,k} - y^{m,k-1}}{h_2} e^{-\beta h_2} + A^{m,k} y^{m,k}_\eta + \\ + \left[B^{m,k} + \nu (w^{m,k} + V_3^{m,k}) V^{m,k}_{3\eta\eta} - \left(\eta \frac{U^{m,k}}{h_2} + \frac{\mu_k}{h_2^{1-\gamma}} \right) (1 - e^{-\beta h_2}) \right] y^{m,k} > 0$$

$$0 < \eta < 1, \quad 1 \leqslant m \leqslant N, \quad 0 \leqslant k \leqslant l, \quad \mu_0 = 0, \quad U^{m,0} = 0, \quad 0 < \gamma < 1 \tag{16}$$

$$[\lambda^1_{m,k}(V_3) - \lambda^1_{m,k}(w)] e^{-\beta k h_2} = \left[\nu y^{m,k}_\eta - \frac{C^{m,k}}{w^{m,k} V_3^{m,k}} y^{m,k} \right] \Bigg|_{\eta=0} > 0$$

$$y^{m,k}(1) = 0, \qquad y^{0,k} = y^{N,k} \tag{17}$$

We set $\mu_k \geqslant 1$ for $k \geqslant 1$ and $\beta = h_2^{-1}$ with h_2 sufficiently small to make the coefficient at $y^{m,k}$ in (16) nonpositive. Let M, K and η_0 be such that $y^{M,K}(\eta_0) \geqslant y^{m,k}(\eta)$, i.e., $y^{M,K}(\eta_0) = \max y^{m,k}(\eta)$. Let us assume that $y^{M,K}(\eta_0) > 0$. Since $y^{0,k} = y^{N,k}$, we can assume $M \geqslant 1$ and $y^{M,K}(\eta_0) \geqslant y^{M-1,K}(\eta_0)$. Since $[C^{m,k}/w^{m,k}V_3^{m,k}(0)>0$, it follows from (17) that $\eta_0 \neq 1$ and $\eta_0 \neq 0$. Hence $0 < \eta_0 < 1$ and the inequalities $y_\eta^{M,K}(\eta_0) = 0$, $y_{\eta\eta}^{M,K}(\eta_0) \leqslant 0$ are valid.

Since the coefficient at $y^{m,0}$ which is equal $\nu(w^{m,0} + V_3^{m,0})V_{3\eta\eta}^{m,0}$, is nonpositive, it follows from inequality (16) that $K \neq 0$ and $y^{M,K}(\eta_0) \geqslant y^{M,K-1}(\eta_0)$. From (16) we then obtain

$$\left[B^{M,K} + \nu(w^{M,K} + V_3^{M,K})V_{3\eta\eta}^{M,K} - \left(\eta\frac{U^{M,K}}{h_2} + \frac{\mu_K}{h_2^{1-\gamma}}\right)(1 - e^{-\beta h_2})\right] y^{M,K}\Big|_{\eta=\eta_0} > 0$$

This inequality is, however, impossible, since the coefficient at $y^{M,K}$ by virtue of selecting nonpositive μ_k, β and h_2. Hence $y^{m,k}(\eta) \leqslant 0$ and the estimate

$$w^{m,k}(\eta) \geqslant M_6(1-\eta)\sigma e^{-\alpha k h_2} \geqslant M_5(1-\eta)\sigma, \quad (M_5 \leqslant M_6 e^{-\alpha X})$$

is valid. Lemma 2 is proved.

Let us determine $w^{m,k}$ for any integral m: $w^{Np+q,k} = w^{q,k}$ $(0 \leqslant q \leqslant N-1)$. In particular, $w^{-1,k} = w^{N-1,k}$. Note that the periodicity T of the coefficients of system (1) and conditions (2) imply that

$$L_{Np+q,k}(w) \equiv L_{q,k}(w) = 0, \qquad \lambda_{Np+q,k}(w) \equiv \lambda_{q,k}(w) = 0$$

We introduce the following notation:

$$r^{m,k} = \frac{w^{m,k} - w^{m,k-1}}{h_2}, \quad z^{m,k} = w_\eta^{m,k}, \quad \rho^{m,k} = \frac{w^{m,k} - w^{m-1,k}}{h_1}$$

Let us establish the uniform with respect to h_i $(i = 1, 2)$ estimates for $r^{m,k}$, $z^{m,k}$ and $\rho^{m,k}$, and write the equations which satisfy these. Differentiating (6) with respect to η, for $z^{m,k}$ we obtain

$$\begin{gathered} P_{m,k}(z) \equiv \nu(w^{m,k})^2 z_{\eta\eta}^{m,k} - \frac{z^{m,k} - z^{m-1,k}}{h_1} - \\ -(\eta U^{m,k} + \mu_k h_2^\gamma)\frac{z^{m,k} - z^{m,k-1}}{h_2} + A^{m,k}z_\eta^{m,k} + (B^{m,k} + A_\eta^{m,k})z^{m,k} + \\ + 2\nu w^{m,k}z^{m,k}z_\eta - U^{m,k}r^{m,k} + B_\eta^{m,k}w^{m,k} = 0 \qquad (18) \\ 1 \leqslant m \leqslant N, \quad 0 \leqslant k \leqslant l, \quad U^{m,0} = 0, \quad \mu_0 = 0, \quad \mu_k > 0 \ (k \geqslant 1), \quad 0 < \gamma < 1 \end{gathered}$$

with conditions

$$z^{m,k}|_{\eta=0} = \frac{1}{\nu}\left(v_0^{m,k} - \frac{C^{m,k}}{w^{m,k}}\right)\Big|_{\eta=0}, \qquad z^{0,k} = z^{N,k} \qquad (19)$$

Subtracting from the equation for $w^{m,k}$ in (6) that for $w^{m,k-1}$ and dividing the difference by h_2, we obtain

$$\begin{gathered} R_{m,k}(r) \equiv \nu(w^{m,k})^2 r_{\eta\eta}^{m,k} - \frac{r^{m,k} - r^{m-1,k}}{h_1} - (\eta U^{m,k-1} + \mu_{k-1}h_2^\gamma) \times \\ \times \frac{r^{m,k} - r^{m,k-1}}{h_2} + A^{m,k}r_\eta^{m,k} + B^{m,k}r^{m,k} + \\ + \frac{w^{m,k} + w^{m,k-1}}{(w^{m,k-1})^2}[\rho^{m,k-1} + (\eta U^{m,k-1} + \mu_{k-1}h_2^\gamma)r^{m,k-1} - A^{m,k-1}z^{m,k-1} - \\ - B^{m,k-1}w^{m,k-1}]r^{m,k} - \left[\eta\frac{U^{m,k} - U^{m,k-1}}{h_2} + \frac{\mu_k - \mu_{k-1}}{h_2^{1-\gamma}}\right]r^{m,k} + \end{gathered}$$

$$+\frac{1}{h_2}(A^{m,k}-A^{m,k-1})z^{m,k-1}+\frac{1}{h_2}(B^{m,k}-B^{m,k-1})w^{m,k-1}=0 \tag{20}$$
$$1\leqslant m\leqslant N,\quad 1\leqslant k\leqslant l,\quad U^{m,0}=0,\quad \mu_0=0,\quad \mu_k\geqslant\mu_{k-1},\quad 0<\gamma<1$$

From conditions (7) and (8) we similarly obtain

$$r^{m,k}(1)=0,\quad \Upsilon_{m,k}(r)\equiv\left[\nu r_\eta^{m,k}-\frac{C^{m,k}}{w^{m,k}w^{m,k-1}}r^{m,k}-\right.$$
$$\left.\left.-\frac{v_0^{m,k}-v_0^{m,k-1}}{h_2}+\frac{C^{m,k}-C^{m,k-1}}{h_2w^{m,k-1}}\right]\right|_{\eta=0}=0,\quad r^{0,k}=r^{N,k} \tag{21}$$

Functions $r^{m,0}$ are undetermined (we can, however, assume that $w^{m,-1}\equiv w^{m,0}$ and, consequently, $r^{m,0}\equiv 0$). Similarly the analysis of equalities

$$\frac{1}{h_1}[L_{m,k}(w)-L_{m-1,k}(w)]=0,\quad \frac{1}{h_1}[\lambda^1_{m,k}(w)-\lambda^1_{m-1,k}(w)]=0$$

yields

$$T_{m,k}(\rho)\equiv\nu(w^{m,k})^2\rho_{\eta\eta}^{m,k}-\frac{\rho^{m,k}-\rho^{m-1,k}}{h_1}-$$
$$-(\eta U^{m,k}+\mu_kh_2^\gamma)\frac{\rho^{m,k}-\rho^{m,k-1}}{h_2}+A^{m,k}\rho_\eta^{m,k}+B^{m,k}\rho^{m,k}+$$
$$+\frac{w^{m,k}+w^{m-1,k}}{(w^{m-1,k})^2}[\rho^{m-1,k}+(\eta U^{m-1,k}+\mu_kh_2^\gamma)r^{m-1,k}-A^{m-1,k}z^{m-1,k}-$$
$$-B^{m-1,k}w^{m-1,k}]\rho^{m,k}-\eta\frac{U^{m,k}-U^{m-1,k}}{h_1}r^{m-1,k}+$$
$$+\frac{1}{h_1}(A^{m,k}-A^{m-1,k})z^{m-1,k}+\frac{1}{h_1}(B^{m,k}-B^{m-1,k})w^{m-1,k}=0 \tag{22}$$
$$1\leqslant m\leqslant N,\quad 0\leqslant k\leqslant l,\quad U^{m,0}=0,\quad \mu_0=0,\quad \mu_k>0\quad (k\geqslant 1),\quad 0<\gamma<1$$

with conditions

$$\rho^{m,k}(1)=0,\quad \Gamma_{m,k}(\rho)\equiv\left[\nu\rho_\eta^{m,k}-\frac{C^{m,k}}{w^{m,k}w^{m-1,k}}\rho^{m,k}-\right.$$
$$\left.\left.-\frac{v_0^{m,k}-v_0^{m-1,k}}{h_1}+\frac{C^{m,k}-C^{m-1,k}}{h_1w^{m-1,k}}\right]\right|_{\eta=0}=0,\quad \rho^{0,k}=\rho^{N,k} \tag{23}$$

($\rho^{0,k}$ are determined, since $w^{-1,k}\equiv w^{N-1,k}$). We assume henceforth that $(U_t/U)|_{\xi=0}=0$, $U_{xt}|_{\xi=0}=0$ and $v_{0t}|_{\xi=0}=0$. From (22) and (23) we then find that $\rho^{m,0}$ must satisfy equations

$$T_{m,0}(\rho)\equiv\nu(w^{m,0})^2\rho_{\eta\eta}^{m,0}-\frac{\rho^{m,0}-\rho^{m-1,0}}{h_1}+A^{m,0}\rho_\eta^{m,0}+$$
$$+\frac{w^{m,0}+w^{m-1,0}}{(w^{m-1,0})^2}[\rho^{m-1,0}-A^{m-1,0}z^{m-1,0}]\rho^{m,0}=0 \tag{24}$$

and conditions

$$\rho^{m,0}(1)=0,\quad \Gamma_{m,0}(\rho)\equiv\left.\left[\nu\rho_\eta^{m,0}-\frac{C^{m,0}}{w^{m,0}w^{m-1,0}}\rho^{m,0}\right]\right|_{\eta=0}=0,\quad \rho^{0,0}=\rho^{N,0} \tag{25}$$

Function $\rho^{m,0}(\eta)\equiv 0$ satisfies Eqs. (24) and conditions (25). Let us consider the solution $w^{m,k}$ of problem (6) – (8) in which $\rho^{m,0}\equiv 0$.

Lemma 3. Let $v_0|_{\xi=0}\leqslant 0$. Then for $0\leqslant\eta<1$, M_7 and $M_8=\text{const}>0$ the estimate

$$- M_7\sigma \leqslant z^{m,0} \leqslant - M_8\sigma \tag{26}$$

is valid for $z^{m,0} = w_\eta^{m,0}$.

Proof. Since $\rho^{m,0} \equiv 0$ and $w^{m,0}$ are independent of m, Lemmas 4 and 5 in [1], from which follows the estimate (26), are valid for $w^{m,0}(\eta) = w^c(\eta)$. Lemma 3 is proved.

Lemma 4. Let

$$v_0 \leqslant E_5\xi, \quad v_{0t} \geqslant - E_6\xi, \quad |U_t / U| \leqslant E_7\xi$$
$$|(U_t/U)_t| \leqslant E_8\xi, \quad U_{xt} \leqslant E_9\xi, \quad |(r_x U r^{-1} - U_x)_t| \leqslant E_{10}\xi \tag{27}$$

It is then possible to find such a positive X_1, which depends on the input data of problem (1), (2), that for $0 \leqslant kh_2 \leqslant X_1$ and $0 \leqslant mh_1 \leqslant T$ the estimates

$$- M_9\sigma \leqslant w_\eta^{m,k} \leqslant - M_{10}\sigma \tag{28}$$

$$\left| \frac{w^{m,k} - w^{m,k-1}}{h_2} \right| \leqslant M_{11}(1-\eta)\sigma \tag{29}$$

$$- M_{12}(1-\eta)\sigma \leqslant \frac{w^{m,k} - w^{m-1,k}}{h_1} \leqslant M_{13}kh_2(1-\eta)\sigma \tag{30}$$

$$|w^{m,k} w_{\eta\eta}^{m,k}| \leqslant M_{14}, \qquad w^{m,k} w_{\eta\eta}^{m,k} \leqslant - M_{15} \tag{31}$$

where M_i are independent of h_1 and h_2 are valid for the solution of problem (6)–(8).

Proof. This is carried out by the method of induction with respect to k. For $k = 0$ the inequalities (28)–(30) are valid and $r^{m,0}$ is undetermined (it will be shown in the proof that the value of $r^{m,0}$ is immaterial).

Let $\Psi^{m,k} = M_{16}w^{m,k}$, $\Phi_1^{m,k} = - M_9\sigma$, $\Phi_2^{m,k} = - M_{10}\sigma$, $F_1^{m,k} = - M_{17}w^{m,k}$ and $F_2^{m,k} = M_{18}kh_2w^{m,k}$. For proving the lemma it is sufficient to establish the validity of the following inequalities:

$$|r^{m,k}| \leqslant \Psi^{m,k}, \qquad \Phi_1^{m,k} \leqslant z^{m,k} \leqslant \Phi_2^{m,k}, \qquad F_1^{m,k} \leqslant \rho^{m,k} \leqslant F_2^{m,k}$$

In fact, if we select $M_{11} \geqslant M_{16}M_2$, $M_{12} \geqslant M_{17}M_2$ and $M_{13} \geqslant M_{18}M_2$, the estimates (28)–(30) are valid. Estimates (31) follow from estimates (9), (15), and (28)–(30) and Eqs. (6).

The proof of Lemma 4 follows very closely that of Lemma 9 in [1]. The important difference is in that here the induction is only with respect to k. This imposes a very strict sequence for proving the estimates. First we estimate $r^{m,k}$, then $z^{m,k}$ and, finally, $\rho^{m,k}$. Furthermore, it should be noted that Eq. (22) can no longer be considered as linear, since by the definition of induction $\rho^{m-1,k}$ has no estimate and $(w^{m,k} + w^{m-1,k})(w^{m-1,k})^{-2}\rho^{m-1,k}\rho^{m,k}$ is a nonlinear term. In the proof of the estimate for $\rho^{m,k}$ we specifically use unequal steps with respect to τ and ξ, i.e., $h_1 \neq h_2$.

Let $R_{m,k}^1$ be the uniform part of operator $R_{m,k}$. From (20) we have

$$R_{m,k}^1(r) + \frac{1}{h_2}(A^{m,k} - A^{m,k-1})z^{m,k-1} + \frac{1}{h_2}(B^{m,k} - B^{m,k-1})w^{m,k-1} = 0$$

Note that the coefficient at $r^{m,0}$ vanishes when $k = 1$. Let us prove that for $0 \leqslant \eta < 1$

$$R_{m,k}^{\sim}(\Psi) \equiv R_{m,k}^1(\Psi) + \frac{1}{h_2}|(A^{m,k} - A^{m,k-1})z^{m,k-1} + (B^{m,k} - B^{m,k-1})w^{m,k-1}| < 0$$

We select $M_{16}(M_2, M_5, M_{10})$ and $x^1(M_2, M_{10}, M_{16}, M_{18})$ so as to satisfy for $kh_2 \leqslant x^1$ and sufficiently small h_2 the following inequalities:

$$M_{16}\,\frac{U^{m,k}-U^{m,k-1}}{h_2} > \frac{1}{h_2}\left|\left(r_x\frac{U}{r}-U_x\right)^{m,k}-\left(r_x\frac{U}{r}-U_x\right)^{m,k-1}\right|$$

$$\frac{1}{2}M_{16}w^{m,k}\,\frac{w^{m,k}+w^{m,k-1}}{(w^{m,k-1})^2}\,A^{m,k-1}z^{m,k-1} \geqslant \left|\frac{A^{m,k}-A^{m,k-1}}{h_2}\,z^{m,k-1}\right| + \frac{1}{h_2}\left|\left(\frac{U_t}{U}\right)^{m,k}-\left(\frac{U_t}{U}\right)^{m,k-1}\right| w^{m,k-1} \tag{32}$$

$$[M_{18}(k-1)h_2+(\eta U^{m,k-1}+\mu_{k-1}h_2^{\gamma})-B^{m,k-1}]\,w^{m,k-1} \leqslant {}^{1}\!/_{2}M_{10}(C^{m,k}+\eta U_x^{m,k})\,(1-\eta)\,\sigma$$

These inequalities are possible, as can be seen from the estimates $U^{m,k}\leqslant N_1kh_2$ and $B^{m,k}|\leqslant N_2kh_2$. It can be shown by the calculation of $\overset{\circ}{R}_{m,k}(\Psi)$ that the required inequality $\overset{\circ}{R}_{m,k}(\Psi)<0$ is the consequence of (32).

Denoting by $\gamma^1_{m,k}$ the uniform part of operator $\gamma_{m,k}$, we obtain

$$\gamma^1_{m,k}(\Psi)+\left|-\frac{v_0^{m,k}-v_0^{m,k-1}}{h_2}+\frac{C^{m,k}-C^{m,k-1}}{h_2w^{m,k-1}(0)}\right| \leqslant$$

$$\leqslant\left\{M_{16}\left[v_0^{m,k}-\frac{C^{m,k}}{w^{m,k}}-\frac{C^{m,k}}{w^{m,k-1}}\right]+\sup|v_{0x}|+\frac{\sup|C_x|}{w^{m,k-1}}\right\}\Bigg|_{\eta=0}<0$$

provided that M_{16} is sufficiently great and $kh_2\leqslant x^2(E_5,M_2)$. Let us examine functions $q_{\pm}^{m,k}=\pm r^{m,k}-\Psi^{m,k}$. It follows from the immediately preceding inequalities and from Eqs. (20) and (21) that

$$R^1_{m,k}(q_{\pm})>0,\qquad \gamma^1_{m,k}(q_{\pm})>0,\qquad q_{\pm}^{m,k}(1)=0$$

$$q_{\pm}^{0,k}=q_{\pm}^{N,k},\qquad q_{\pm}^{m,k-1}\leqslant 0$$

Using these relationships and repeating the reasoning of Lemma 2, we conclude that $q_{\pm}^{m,k}\leqslant 0$, hence the estimate $-\Psi^{m,k}\leqslant r^{m,k}\leqslant\Psi^{m,k}$ is valid for $kh_2\leqslant\xi^1=\min(x^1,x^2)$.

The estimate for $z^{m,k}$ is derived in exactly the same manner as that for z^k in Lemma 6 in [1]. Omitting the proof, we would only mention that M_9 depends on M_{16} and M_{10} on the input data of the problem (1), (2) and constants M_2 and M_5 (it is important that M_{10} is independent of M_{16}); here we also impose the condition of smallness of X_1: $kh_2\leqslant x^3(M_{10},M_{16},M_2,E_5)$.

We pass to the estimate of $\rho^{m,k}$. It can be verified that for $kh_2\leqslant x^2(E_5,M_2)$ and sufficiently large $M_{17}(M_{16},M_2,M_5)$ the inequalities $T_{m,k}(F_1)>0\;(0\leqslant\eta<1)$ and $\Gamma_{m,k}(F_1)>0$ are valid. By calculating $T_{m,k}(F_2)$ and $\Gamma_{m,k}(F_2)$ it can be readily shown that $T_{m,k}(F_2)<0\quad(0\leqslant\eta<1)$ and $\Gamma_{m,k}(F_2)<0$, when $kh_2\leqslant\min\{x^2,x^4(M_{16},M_{18},M_{10})\}$ and $M_{18}(M_{16},E_6,E_8,E_9)$ have been made sufficiently great and $h_1,h_2\leqslant h_0$. This is, in fact, possible provided that the following inequalities are satisfied:

$$M_{18}U^{m,k}w^{m,k-1}\geqslant\frac{1}{h_1}\left|\left(r_x\frac{U}{r}-U_x\right)^{m,k}-\left(r_x\frac{U}{r}-U_x\right)^{m-1,k}\right|w^{m-1,k}+ + M_{16}\left|\frac{U^{m,k}-U^{m-1,k}}{h_1}\right|w^{m-1,k}$$

$$\frac{1}{2}M_{18}kh_2\,\frac{w^{m,k}(w^{m,k}+w^{m-1,k})}{(w^{m-1,k})^2}\,A^{m-1,k}z^{m-1,k}\geqslant$$

$$\geqslant\frac{A^{m,k}-A^{m-1,k}}{h_1}\,z^{m-1,k}+\frac{1}{h_1}\left|\left(\frac{U_t}{U}\right)^{m,k}-\left(\frac{U_t}{U}\right)^{m-1,k}\right|w^{m-1,k}$$

$$(M_{18}kh_2 + M_{16}(\eta U^{m-1,k} + \mu_k h_2^{\gamma}) - B^{m-1,k})\, w^{m-1,k} \leqslant$$
$$\leqslant {}^1/_2 A^{m-1,k} z^{m-1,k} \leqslant {}^1/_2 M_{10}(C^{m-1,k} + \eta U_x^{m-1,k})(1-\eta)\sigma$$

These inequalities follow from the conditions of this lemma, smoothness of coefficients and the independence of M_{16} of M_{18}.

Let us examine the remainders $S_1^{m,k} = F_1^{m,k} - \rho^{m,k}$ and $S_2^{m,k} = \rho^{m,k} - F_2^{m,k}$. By definition of induction $S_j^{m,k-1} \leqslant 0$ $(j=1,2)$. For $S_j^{m,k}$ we have

$$\nu (w^{m,k})^2 S_{j\eta\eta}^{m,k} - \frac{S_j^{m,k} - S_j^{m-1,k}}{h_1}\left[1 + h_1 \frac{w^{m,k} + w^{m-1,k}}{(w^{m-1,k})^2} F_j^{m,k}\right] +$$
$$+ A^{m,k} S_{j\eta}^{m,k} + \left[B_1^{m,k} + \frac{w^{m,k} + w^{m-1,k}}{(w^{m-1,k})^2}\rho^{m-1,k} - \right.$$
$$\left. - \left(\eta \frac{U^{m,k}}{h_2} + \frac{\mu_k}{h_2^{1-\gamma}}\right)\right] S_j^{m,k} > 0 \tag{33}$$

$$B_1^{m,k} = B^{m,k} + \frac{w^{m,k} + w^{m-1,k}}{(w^{m-1,k})^2}[F_j^{m,k} + (\eta U^{m-1,k} + \mu_k h_2^{\gamma})\, r^{m-1,k} -$$
$$- A^{m-1,k} z^{m-1,k} - B^{m-1,k} w^{m-1,k}]$$

$$\left[\nu S_{j\eta}^{m,k} - \frac{C^{m,k}}{w^{m,k} w^{m-1,k}} S_j^{m,k}\right]\Bigg|_{\eta=0} > 0,\ S_j^{m,k}(1) = 0,\ S_j^{0,k} = S_j^{N,k} \tag{34}$$

Let us prove that for certain relationships between the steps h_1 and h_2 the coefficient at $S_j^{m,k}$ in the inequality (33) can be made negative. Using the principle of maximum (see Lemma 2), from (33) and (34) we obtain $S_j^{m,k} \leqslant 0$ $(j = 1, 2)$ and, consequently, the estimates $F_1^{m,k} \leqslant \rho^{m,k} \leqslant F_2^{m,k}$ if $kh_2 \leqslant X_1 = \min(x^i)$ and h_1 and h_2 are sufficiently small. This will serve to prove Lemma 4. Let us specify $\mu_k \geqslant 1$ with $k \geqslant 1$ and h_2 so small that

$$B_1^{m,k} - \left(\eta \frac{U^{m,k}}{h_2} + \frac{1}{2}\frac{1}{h_2^{1-\gamma}}\right) < 0 \tag{35}$$

which is possible, since $\gamma < 1$ and the estimates for $r^{m-1,k}$ and $z^{m-1,k}$ have been already proved. Furthermore,

$$\frac{w^{m,k} + w^{m-1,k}}{(w^{m-1,k})^2}\rho^{m-1,k} - \frac{\mu_k - {}^1/_2}{h_2^{1-\gamma}} \leqslant \left(\frac{w^{m,k} + w^{m-1,k}}{w^{m-1,k}}\right)^2 \frac{1}{h_1} - \frac{\mu_k - {}^1/_2}{h_2^{1-\gamma}} \leqslant 0 \tag{36}$$

if $\mu_k \geqslant {}^1/_2 + (1 + M_2/M_5)^2$ and $h_1 \geqslant h_2^{1-\gamma}$. Lemma 4 is proved.

Theorem 1. Let the propositions of Lemma 4 be satisfied by the T-periodic functions r, U and v_0. Then there exists in Ω_{X_1} $\{0 \leqslant \tau \leqslant T,\ 0 \leqslant \xi \leqslant X_1,\ 0 \leqslant \eta < 1\}$ (X_1 is a certain number dependent on U, r and v_0) a solution of problem (4), (5) which has the following properties: $w(\tau, \xi, \eta)$ is continuous in Ω_{X_1}, $M_5(1-\eta)\sigma \leqslant w \leqslant M_2(1-\eta)\sigma$, w_η is continuous with respect to η for $\eta < 1$, $-M_9\sigma \leqslant w_\eta \leqslant -M_{10}\sigma$, w_ξ, w_τ and $ww_{\eta\eta}$ are bounded in Ω_{X_1} and

$$|w_\xi| \leqslant M_{11}(1-\eta)\sigma,$$
$$ww_{\eta\eta} \leqslant -M_{15}, \quad -M_{12}(1-\eta)\sigma \leqslant w_\tau \leqslant M_{13}\xi(1-\eta)\sigma.$$

Function w satisfies Eq. (4) almost everywhere, and for $0 \leqslant \xi \leqslant X_1$ conditions (5) are satisfied. The solution of problem (4), (5) which has these properties is unique in $\Omega_{X_2} \subset \Omega_{X_1}$.

The Proof of existence follows the reasoning of Theorems 7 and 11 in [1]. We shall prove the uniqueness of this solution of problem (4), (5). Let w_1 and w_2 be two solutions of this problem and $w_* = w_1 - w_2$. For w_* we obtain

$$\nu w_1 w_{*\eta\eta} - \frac{w_{*\tau}}{w_1} - \eta U \frac{w_{*\xi}}{w_1} + A\frac{w_{*\eta}}{w_1} + B\frac{w_*}{w_1} + \nu(w_1+w_2)\, w_{2\eta\eta}\frac{w_*}{w_1} = 0 \qquad (37)$$

$$\left(\nu w_{*\eta} - C\frac{w_*}{w_1 w_2}\right)\Big|_{\eta=0} = 0,\; w_*|_{\eta=1} = 0,\; w_*|_{\tau=0} = w_*|_{\tau=T} \qquad (38)$$

Multiplying Eq. (37) by $w_* e^{-\alpha\xi}$ and integrating with respect to Ω_{X_2} ($\alpha = \text{const} > 0$), we transform certain terms of this equality by integration by parts and obtain

$$\int\limits_{\Omega_{X_2}} [-\nu w_1 (w_{*\eta})^2]\, e^{-\alpha\xi} d\tau d\xi d\eta + \int\limits_{\xi=X_2} \left[-\eta U \frac{w_*^2}{2w_1}\right] e^{-\alpha\xi} d\tau d\eta +$$
$$+ \int\limits_{\Omega_{X_2}} \frac{1}{2w_1^2}\left[\nu w_1^2 w_{1\eta\eta} - w_{1\tau} - \eta U w_{1\xi} + A w_{1\eta} + B w_1\right] w_*^2 e^{-\alpha\xi} d\tau d\xi d\eta +$$
$$+ \int\limits_{\Omega_{X_2}} \frac{1}{w_1}\left[-\frac{1}{2}\alpha\eta U - \frac{1}{2}\eta U_\nu + \frac{1}{2}\eta\left(r_x\frac{U}{r} - U_x\right) - \frac{U_t}{U} + \right.$$
$$\left. + \nu(w_1+w_2)\, w_{2\eta\eta}\right] w_*^2 e^{-\alpha\xi} d\tau d\xi d\eta +$$
$$+ \int\limits_{\eta=0} \left[-\frac{C}{w_2} + \frac{1}{2}\nu w_{1\eta} - A\frac{1}{2w_1}\right] w_*^2 e^{-\alpha\xi} d\tau d\xi = 0 \qquad (39)$$

We have used here the conditions (38) and the condition $U|_{\xi=0} = 0$. We select α from the inequality $|r_x U r^{-1} - U_x| \leqslant \alpha U$ and chose X_2 so small that

$$\left[-\frac{C}{w_2} + \frac{1}{2}\left(\nu w_{1\eta} - A\frac{1}{w_1}\right)\right]\Big|_{\eta=0} = \left(-\frac{C}{w_2} + \frac{1}{2}v_0\right)\Big|_{\eta=0} \leqslant 0,\; -\frac{U_t}{U} + \nu(w_1+w_2)_{w_{2\eta\eta}} \leqslant 0$$

The left-hand part of equality (39) represents then the sum of integrals of positive functions, which implies that each of these integrals is equal zero. Since in the integral taken over region Ω_{X_1} the coefficient at w_*^2 is negative, $w_* = 0$ almost everywhere. The continuity of w_* in Ω_{X_2} implies that $w_* \equiv 0$ and $w_1 \equiv w_2$. Theorem 1 is proved. We can easily show that $\Omega_{X_2} = \Omega_{X_1}$.

Theorem 2. Let $U(t, x) = ax + b(t, x)x^2$ with $a = \text{const} > 0$ and $b = b(t, x)$ have bounded second-order derivatives, $v_0 \leqslant K_1 x$, $v_{0t} \geqslant -K_2 x$, $r_x|_{x=0} > 0$, $|(r_x U r^{-1} - U_x)_t| \leqslant K_3 x$, and b, v_0 and r are periodic functions with respect to t of period T. There exists then a solution u, v of the problem (1), (2) which is unique in D_{X_2} $\{0 \leqslant t \leqslant T,\ 0 \leqslant x \leqslant X_2,\ 0 \leqslant y \leqslant \infty\}$ and has the following properties: u/U and u_y/U are bounded and continuous in D_{X_2}, $u > 0$ for $y > 0$ and $x > 0$; $u \to U$ for $y \to \infty$, $u|_{x=0} = 0$, $u|_{y=0} = 0$, $u_y/U > 0$ for $y \geqslant 0$ and $u_y/U \to 0$ for $y \to \infty$; u_y, u_x, u_{yy}, u_t and v_y are bounded and continuous with respect to y; u_{yy}/u_y and v are continuous with respect to y and bounded for finite y, $v|_{y=0} = v_0(t, x)$ and u_{yyy} is bounded in D_{X_2}, u_{yx} and u_{yt} are bounded for finite y. The equations of system (1) are satisfied almost everywhere in D_{X_2}. Furthermore the inequalities

$$K_4 (U-u)\sigma \leqslant u \leqslant K_5 (U-u)\sigma$$

$$-K_6\sigma \leqslant \frac{u_{yy}}{u_y} \leqslant -K_7\sigma, \qquad \left|\frac{u_{yyy}u_y - u_{yy}^2}{u_y^2}\right| \leqslant K_8$$

$$\left|\frac{1}{u_y}(u_{yx}u_y - u_x u_{yy}) + \frac{U_x}{u_y U}(uu_{yy} - u_y^2)\right| \leqslant K_9 (U-u)\sigma$$

$$-K_{10}(U-u)\sigma \leqslant \frac{1}{u_y}(u_{yt}u_y - u_t u_{yy}) + \frac{U_t}{u_y U}(uu_{yy} - u_y^2) \leqslant K_{11}x(U-u)\sigma$$

are satisfied. In the above inequalities $\sigma = [-\ln \mu\ (1 - u / U)]^{1/2}$, K_i and μ are certain positive constants, $0 < \mu < 1$. and $X_2 > 0$ depends on U, r and v_0.

This theorem is the corollary of Theorem 1.

We note in conclusion that the stipulations and the input data of problem (1), (2) formulated in Lemma 4 and Theorems 1 and 2 are somewhat less stringent than the limitations imposed in [1]. The analysis presented here has to a certain extent improved the results obtained in [1] and made it possible to prove the theorem of existence of solution of the Cauchy problem for the requirements with respect to external flow, as specified in Theorems 1 and 2.

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BIBLIOGRAPHY

1. Oleinik, O. A., Mathematical problems of the boundary layer theory. Uspekhi Matem. Nauk, Vol.23, №4, 1968.
2. Liusternik, L. A. and Sobolev, V. P., Elements of Functional Analysis, p.291, "Nauka", Moscow, 1965.

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ON THE PLANE-PARALLEL SYMMETRIC BOUNDARY LAYER GENERATED BY SUDDEN MOTION

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The formation of a boundary layer over a body which suddenly begins to move in a stationary incompressible fluid is analyzed. Proof is given of the existence and uniqueness under certain conditions of solution of the related boundary value problem defined by the system of Prandtl's equations in a certain time interval $0 \leqslant t \leqslant T$ and over the whole of the streamlined body. This problem was also considered by Blasius [1] who had proposed to solve it by expanding the stream function into an asymptotic series in powers of time, and had given the first two terms of this expansion in their explicit form. A brief account of these results and the mathematical formulation of the problem appear in [2, 3]. The problem of boundary layer development under conditions of gradual acceleration was